Mixed Use of Pontryagin's Principle and the Hamilton-Jacobi-Bellman Equation in Infinite- and Finite-Horizon Constrained Optimal Control

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Abstract. This paper proposes a framework for solving a class of nonlinear infinite- and finite-horizon optimal control problems with constraints. Establishment of existence and uniqueness of solutions to the Hamilton-Jacobi-Bellman (HJB) equation plays a crucial role in verifying well-posedness of a given problem and in streamlining numerical solutions. The proposed framework revolves around infinite-horizon Bolza-type cost functions with running costs exponentially decaying in time. We show Γ-convergence of solutions with such cost functions to the solutions of initial constrained (in)finite-horizon problems (that is, without running costs exponentially decaying in time). Basically, we demonstrate how to approximate solutions of (in)finite-horizon constrained optimal problems using our framework. Employing a solver based on the Pontryagin's Principle, we efficiently obtain optimal solutions for finite- and infinite-horizon problems. Efficiency of the proposed framework is demonstrated in simulation by solving a 3D path planing problem with obstacles for a full nonlinear model of an autonomous underwater vehicle (AUV).

1 INTRODUCTION

Building upon recent existence and uniqueness results regarding the Hamilton-Jacobi-Bellman (HJB) equation [2], this paper proposes a novel framework for solving a rather general class of infinite- and finite-horizon optimal control problems with constraints. The class of optimal problems considered herein includes nonlinear affine-in-control system dynamics and performance indices with locally bounded, Lipchitz and convex integral functions. Elliptic, and therefore convex, state constraints are considered and handled via the Valentine transformation. In addition, the admissible control signals belong to time-varying convex sets. We utilize the off-the-shelf software package `DIDO` [17], built upon the pseudo-spectral analysis and Pontryagin's Principle (PP), to numerically obtain the unique trajectories/controls starting from various initial positions. In a nutshell, given a constrained optimal control problem, we extend earlier HJB results devised for unconstrained problems and make a connection with PP to facilitate the application of PP-based solvers.

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The proposed framework revolves around infinite-horizon Bolza-type cost functions with running costs exponentially decaying in time. This class of integral functions leads to existence and uniqueness of the HJB solutions [2], and mitigates the HJB solution-seeking process. We show Γ-convergence of solutions with these cost functions to the solutions of the original constrained problems whose running costs do not have to decay exponentially in time. Essentially, we demonstrate how to employ our infinite-horizon framework to approximate the original constrained optimal problems.

Prevalent approaches towards solving HJB are found in, among others, [1], [2], [13], [8], [12]. Owing to recent advances in computational power, a large portion of nowadays approaches is based on Dynamic Programming (DP) such as Approximate Dynamic Programming (ADP) and Reinforcement Learning (RL) [7]. ADP refers to model-based approaches whilst RL refers to model-free approaches. It is to be noted that the approach delineated herein is model based. However, ADP and RL do not deal well with constraints [12], but so-called soft constraints are typically imposed through penalty terms in cost functions, which is yet another incentive behind our framework. A somewhat more mature and more akin to our approach is Model Predictive Control (MPC) [8]. However, optimality of MPC solutions is typically traded for computational efficiency and stability guarantees. The present work aims for maintaining optimality when resorting to numeric solutions of HJB.

Examples of approaches concerning state-constrained optimal control are found in [3], [11], [19]. These approaches concern using PP and vary in terms of focus on measure spaces [3], state constraints at specific timestamps [11], and state-constrained infinite-horizon problems[19]. Compared to these works, our approach first approximates a finite-time, state constrained optimal control with an infinite-horizon version, converts this into an unconstrained problem through a formal transformation, and then applies HJB results to this new problem.

The principal contributions of this paper are fourfold: a) we extend previous results on HJB existence and uniqueness to constrained problems in order to b) prove optimality of solutions to HJB so that we can c) use a solver based on Pontryagin to efficiently compute this optimal control, and d) demonstrate this approach on finite-horizon problems of path planing in 3D with obstacles for the nonlinear AUV dynamics.

The remainder of this paper is organized as follows. Section 2 introduces the utilized notation and definitions. The infinite- and finite-horizon constrained optimal control problems considered herein are formulated in Section 3. Section 4 proposes methodology to successfully solve the problems of interest. Section 5 establishes relations between solutions obtained in Section 4 and the original finite-horizon problems from Section 3.

Section 6 provides numerical examples. The conclusions and future research avenues are in Section 7. Appendix brings several technical results aiding in paper self-containment.

2 PRELIMINARIES

Going forward, we use the following conventions. We denote by $|\cdot|$, $|\cdot|_\infty$, $|\cdot|_m$, and $\langle\cdot,\cdot\rangle$ the Euclidean norm, the max norm, the matrix norm induced by the Euclidean norm, and the inner product in $\mathbb{R}^k$, respectively. For a nonempty set $C \subset \mathbb{R}^k$, we denote its closure by $\overline{C}$ and its boundary by ∂C. For matrix M, we denote its Moore-Penrose (left) inverse by $M^\dagger$.

For row vectors we use $r = [r_1 \; \ldots \; r_{N_x}]$ and for column vectors we use $r = [r_1; \, \ldots; \, r_{N_x}]$, where r_i's are scalars. When the argument of a function $f : \mathbb{R} \to \mathbb{R}$ is a row vector r, it means $f(r) = [f(r_1) \; \ldots \; f(r_{N_x})]$. Likewise, for a column vector r, we have $f(r) = [f(r_1); \, \ldots; \, f(r_{N_x})]$. For example, let row vector $r = [0\ 1\ 2\ 3]$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Then $f(r) = [0\ 1\ 4\ 9]$. Let $I(r)$ be a diagonal matrix where the diagonal entries are given by vector $r \in \mathbb{R}^{N_x}$. Unless stated otherwise, vectors are column vectors by default. Let $\{x^{(l)}\}$ denote a sequence of elements.

Define the matrix $\hat{x} \in SO(3)$ such that for $x \in \mathbb{R}^3$, we have

$$\hat{x} = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}.$$

Let I and J be closed intervals in $\mathbb{R}$. Let μ be the Lebesgue measure. Denote by $L^1(I;\, J)$ the set of all J-valued Lebesgue integrable functions on I. We say that $f \in L^1_{loc}(I;\, J)$ if $f \in L^1(K;\, J)$ for any compact subset $K \subset I$. We denote by $\mathscr{L}_{loc}$ the set of all functions $f \in L^1_{loc}([0,\infty);\, \mathbb{R}^+)$ such that $\lim_{\sigma \to 0} \sup\{\int_J f(\tau)\, d\tau \, : J \subset [0,\infty),\ \mu(J) \leq \sigma\} = 0$. We say that a function is C^1 if it is continuous and has continuous first derivatives in all variables. For a function f, denote $f^{(i)}$, $i \in \mathbb{N}$, as the i^{th} time derivative.

A set-valued function $G : \mathbb{R}^k \rightsquigarrow \mathbb{R}^n$ is *lower semicontinuous* if, for all $x \in \mathbb{R}^k$, $\operatorname*{Lim\,inf}_{y \to x} G(y) \subset G(x)$. Finally, given a family of functions $\{F_n\}$, we say that a function F is the Γ-limit of $\{F_n\}$ (denoted $F_n \xrightarrow{\Gamma} F$) if $F = F^{sup} = F^{inf}$, where $F^{inf}(x) = \inf\{\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \to x\}$, and $F^{sup}(x) = \inf\{\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \to x\}$. Note that $F^{inf} \leq F^{sup}$.

2.1 Motivations for using `DIDO`

In this paper, our main theorem involves establishing existence and uniqueness of controls and viscosity solutions for a HJB equation derived from a given optimal control problem. It is noted that `DIDO` is based on PP and thus it is beneficial to show a relation between PP and HJB.

Consider the method of characteristics for first order nonlinear PDEs. For general first order PDEs of the form

$$\begin{aligned} F(x,V,V_x) = 0 \ , \ x \in \Omega \subset \mathbb{R}^N \\ V(x) = \overline{V}(x) \ , \ x \in \partial\Omega \end{aligned} \tag{1}$$

with a continuous function $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and endpoint data $\overline{V}(x)$, one can construct a solution V by solving the characteristic ODEs (substituting $p = V_x$)

$$\begin{aligned} \dot{x}_i &= F_{p_i}, \ \ i = 1, \cdots, n, \\ \dot{V} &= \sum_i p_i F_{p_i}, \\ \dot{p}_i &= -F_{x_i} - F_V p_i, \end{aligned} \tag{2}$$

with $x(0) = y$, $V(0) = V(y)$, $p(0) = V_x(y)$ for $y \in \partial\Omega$. It can be shown ([13, Section 7.2], [6, Chapter 8]) that the characteristic ODEs for HJB are the canonical ODEs for PP. It should be noted that the underlying derivation is based on sufficient smoothness of all functions involved – including V being C^1. Fortunately, characteristic ODEs can still be used with weaker assumptions on V ([13, Section 7.2], [6, Chapter 8]).

A side motivation to connect PP and HJB is the following. Controls generated by PP are, in general, *open-loop*, that is, $u = u(t)$. Controls given by HJB, in contrast, are *closed-loop*, that is, $u = u(t,x)$. Consequently, the concept of real-time optimal control (RTOC) can be used to transform an open-loop control, such as those constructed by `DIDO`, into a closed-loop control. The idea behind RTOC is to assume that an open-loop control exists beforehand and then to account for non-zero computational time of the optimal trajectory (i.e., updates based on given data). Methods that could be used to transform open-loop controls generated by `DIDO` into closed-loop controls are found in [16], [18].

3 PROBLEM FORMULATION

Before stating the problem, let us define our state space for ease of notation. Let $\mathbb{X}_s \subset \mathbb{R}^{N_s}$, with $N_s \in \{2, 3\}$, be the obstacle space and $\mathbb{X}_{ns} \subset \mathbb{R}^{k*N_s}$, with $k \in \{0, 1\}$, be the free space; for $k = 0$, one has $\mathbb{X}_{ns} = \emptyset$. Let $\mathbb{X} = \mathbb{X}_s \times \mathbb{X}_{ns}$ with both $\mathbb{X}_s$ and $\mathbb{X}_{ns}$ as open, connected, and bounded sets whose cluster points fall within $\mathbb{X}$. We take $\partial\mathbb{X}$ to be sufficiently smooth almost everywhere. The dimension of $\mathbb{X}$ is equal to $N_x = (k+1)N_s$. Here $\mathbb{X}_s$ represents the spatial region in which an autonomous system moves – typically in 2- or 3-dimensions. $\mathbb{X}_{ns}$ represents the space in which all other variables (i.e., velocities, rotations, etc) are active.

We consider the following optimal control problem:

$$\begin{aligned} & x \in \mathbb{X} \subset \mathbb{R}^{N_x} \ \ , \ u(t) \in U(t) \subset \mathbb{R}^{N_u} \\ \min_{(x,u)} \ & J(t_0,t_f,x_f,x,u) = E(x(t_f),x_f) + \textstyle\int_{t_0}^{t_f} l(s,x(s),u(s))ds \end{aligned} \tag{3}$$

subject to

$$\begin{aligned} \dot{x} &= a(t,x) + b(t,x)u, \\ h^L &\le h(x), \\ x(t_0) &= x_0, \end{aligned}$$

where $a : [t_0,t_f] \times \mathbb{X} \to \mathbb{R}^{(k+1)N_s}$ and $b : [t_0,t_f] \times \mathbb{X} \to \mathbb{R}^{(k+1)N_s \times N_u}$ are defined as follows for $x = [x_s;\ x_{ns}]$:

$$a(t,x) = \begin{bmatrix} a_{K_o}(t,x) \\ a_e(t,x) \end{bmatrix},\ b(t,x) = \begin{bmatrix} b_{K_o}(t,x) \\ b_e(t,x) \end{bmatrix} \tag{4}$$

with functions $a_{K_o} : [t_0,t_f] \times \mathbb{X} \to \mathbb{R}^{N_s}$, $a_e : [t_0,t_f] \times \mathbb{X} \to \mathbb{R}^{k*N_s}$, $b_{K_o} : [t_0,t_f] \times \mathbb{X} \to \mathbb{R}^{N_s \times N_u}$, and $b_e : [t_0,t_f] \times \mathbb{X} \to \mathbb{R}^{k*N_s \times N_u}$ bounded, Lipschitz continuous, and convex (componentwise) in $x \in \mathbb{X}$ for almost all $t \ge t_0$. For $k = 0$, the dynamics reduce to $\dot{x} = a_{K_o}(t,x) + b_{K_o}(t,x)u$. It should be noted that many mobile agent models follow this structure. Furthermore, $U(t)$ is a compact and convex subset of $\mathbb{R}^{N_u}$ for each t. Also, we assume the dynamical system $\dot{x} = a(t,x) + b(t,x)u$ is controllable within $\mathbb{X}$ while satisfying all state constraints.

Define $h : \mathbb{X}_s \to \mathbb{R}^{n_o}$ componentwise as $h_i(x_s) = \sum_{j=1}^{N_s} \left(\frac{x_{s,j}(t) - \bar{x}_{i,j}}{r_{i,j}} \right)^2$ for $i = 1, \ldots, n_o$ where $\bar{x}_i = (\bar{x}_{i,1}, \bar{x}_{i,2}, \ldots, \bar{x}_{i,N_s})$, $r_i = (r_{i,1}, r_{i,2}, \ldots, r_{i,N_s})$ represents the center and principal axis lengths of an ellipsoid, respectively, and n_o represents the number of state constraints (that is, obstacles) in our model with $n_o \geq N_s$. (The case for $0 < n_o < N_s$ will appear in future works.) We define h^L componentwise as $h_i^L = (1 + d_i)^2$, where $d_i > 0$ represents a buffer for avoiding obstacles (for example, when autonomous agents are represented merely as points, d_i takes into account their actual dimensions so as to avoid collisions). Without loss of generality, it is also assumed that the obstacles are completely contained in $\mathbb{X}_s$ (just cut out the partially contained obstacles from $\mathbb{X}_s$). For further clarification, the inequality constraint $h^L \leq h(x)$ should be interpreted componentwise as $h_i^L \leq h_i(x)$ for $i = 1, \ldots, n_o$.

Let $l(t,x,u)$ be Lipschitz in x and u, lower semicontinuous in t, bounded for all $t \geq t_0$, $x \in \mathbb{X}$, $u(t) \in U(t)$, convex in x, and strictly convex in u. The terminal cost $E(\cdot)$ for finite-horizon problems is typically positive definite whereas it is absent in infinite-horizon problems. Clearly, we allow t_f to be finite or infinite. In what follows, we seek to approximate a solution to problem (3) in a reasonable way.

4 METHODOLOGY

In order to approximate (3), we exploit the following auxiliary optimal control problem:

$$\begin{aligned} & x \in \mathbb{X} \ , \ u(t) \in U(t) \\ \min_{(x,u)} \ & J_\infty(t_0, x, u, \lambda) = \int_{t_0}^{\infty} l(s, x(s), u(s)) e^{-\lambda s} ds \\ & \text{subject to} \\ & \dot{x} = a(t,x) + b(t,x)u, \\ & h^L \leq h(x) \\ & x(t_0) = x_0, \end{aligned} \tag{5}$$

for $\lambda > 0$.

The approximation of (3) by (5) comes down to the following two factors:

1. How well $\int_{t_0}^{t_f} l(s, x(s), u(s)) e^{-\lambda s} ds$ approximates $\int_{t_0}^{t_f} l(s, x(s), u(s)) ds$ for some $\lambda > 0$ and interested t_f.

2. How well $\int_{t_f}^{\infty} l(s, x(s), u(s)) e^{-\lambda s} ds$ approximates $E(x(t_f), x_f)$ for some $\lambda > 0$ and interested final state x_f.

Further details are covered in Section 5.

4.1 Application to State- Constrained Problem

We can transform (5) into an unconstrained problem using the Valentine transformation [20, 15]. Consider a slack function $\alpha : [t_0, \infty) \to \mathbb{R}^{n_o}$ such that

$$\tilde{h}(x_s(t)) + \frac{1}{2}(\alpha(t))^2 = 0, \tag{6}$$

where $\frac{1}{2}(\alpha(t))^2 = \frac{1}{2}[\alpha_1(t)^2, \ldots, \alpha_{n_o}(t)^2]^T$. We also consider the related differential equation

$$\frac{d}{dt}(\tilde{h}(x_s(t)) + \frac{1}{2}(\alpha(t))^2) = 0,$$

which can be reduced to

$$\nabla(\tilde{h}(x_s(t)))(a_{K_o}(t,x) + b_{K_o}(t,x)u) + I(\alpha(t))\dot{\alpha}(t) = 0, \tag{7}$$

where $I(\alpha)$ is an $n_o \times n_o$ diagonal matrix with the main diagonal equal to α. By the implicit function theorem, we can write the control u from (7) as $u = \Phi(t, x(t), \alpha(t), \dot{\alpha}(t))$ and define a new control $\tilde{u} = \dot{\alpha}$. Thus, we reach the following unconstrained problem

$$\begin{array}{c} (x,\alpha) \in \mathbb{X} \times \mathbb{A}, \;\; \tilde{u} \in \tilde{U}(t,x,\alpha) \subset \mathbb{R}^{n_o} \\ \min\limits_{(x,\tilde{u})} \tilde{J}_\infty(t_0, x, \tilde{u}, \lambda) = \int_{t_0}^{\infty} \tilde{F}_\lambda(s, x(s), \alpha(s), \tilde{u}(s))ds \end{array} \tag{8}$$

such that

$$\begin{aligned} \dot{x} &= a(t,x) + b(t,x)\Phi(t,x,\alpha,\tilde{u}), \\ \dot{\alpha} &= \tilde{u}, \\ x(t_0) &= x_0, \\ \alpha(t_0) &= [\sqrt{-2\tilde{h}_1(x_{s,0})}, \ldots, \sqrt{-2\tilde{h}_{n_o}(x_{s,0})}]^\top, \end{aligned} \tag{9}$$

where $\mathbb{A} \subset \mathbb{R}^{n_o}$ and $\tilde{F}_\lambda(t,x,\alpha,\tilde{u}) := F_\lambda(t,x,\Phi(t,x,\alpha,\tilde{u}))$. We also define the corresponding Hamiltonian $\tilde{H}(t,x,\alpha,\tilde{u},p) := \tilde{F}_\lambda(t,x,\alpha,\tilde{u}) + \langle p, a(t,x) + b(t,x)\Phi(t,x,\alpha,\tilde{u})\rangle$.

Also, for our problem, we are able to give an explicit formula for Φ. From (7), we have

$$\begin{aligned} u = &-[\nabla(\tilde{h}(x_s(t)))b_{K_o}(t,x(t))]^\dagger * \\ &*[\nabla(\tilde{h}(x_s(t)))a_{K_o}(t,x(t)) + I(\alpha(t))\tilde{u}(t)], \\ = &\,\Phi(t,x,\alpha,\tilde{u}). \end{aligned} \tag{10}$$

Guaranteeing existence and uniqueness of $\tilde{u}$ is a major step in proving our main result. To that end, we introduce the following lemma:

Lemma 1. *For a fixed $(x,\alpha) \in \mathbb{X} \times \mathbb{A}$, the control set $\tilde{U}(t,x,\alpha) = \Phi^{-1}(t,x,\alpha,U(t))$ is a compact and convex set for each t.*

Proof. We start by establishing the convexity of $\tilde{U}(t,x,\alpha)$. Note that $\tilde{U}(t,x,\alpha)$ is the preimage of $U(t)$ under Φ. Thus it suffices to show that, for any $\tilde{u}_1(t),\ \tilde{u}_2(t) \in \tilde{U}(t,x,\alpha)$, the following holds for $\lambda \in [0,1]$ and each (x,α):

$$\tilde{u}_\lambda(t) := \lambda\tilde{u}_1(t) + (1-\lambda)\tilde{u}_2(t) \in \tilde{U}(t).$$

Call

$$A_0(t,x) = -[\nabla(\tilde{h}(x_s))b_{K_o}(t,x)]^\dagger * [\nabla(\tilde{h}(x_s))a_{K_o}(t,x)]$$

and

$$B_0(t,x,\alpha) = -[\nabla(\tilde{h}(x_s))b_{K_o}(t,x)]^\dagger * I(\alpha).$$

With this we have $\Phi(t,x,\alpha,\tilde{u}) = A_0(t,x) + B_0(t,x,\alpha)\tilde{u}$. Continuing from this, suppressing (x,α), we have

$$\begin{aligned}
\Phi(t,\tilde{u}_\lambda(t)) &= A_0(t) + B_0(t)(\lambda\tilde{u}_1(t) + (1-\lambda)\tilde{u}_2(t)), \\
&= \lambda A_0(t) + (1-\lambda)A_0(t) \\
&\quad + B_0(t)(\lambda\tilde{u}_1(t) + (1-\lambda)\tilde{u}_2(t)), \\
&= \lambda\Phi(t,\tilde{u}_1(t)) + (1-\lambda)\Phi(t,\tilde{u}_2(t)), \\
&=: \lambda u_1(t) + (1-\lambda)u_2(t) =: u_\lambda.
\end{aligned}$$

Since $U(t)$ is convex, it is clear that $u_\lambda \in U(t)$ and thus $\tilde{u}_\lambda \in \tilde{U}(t)$. Thus $\tilde{U}(t,x,\alpha)$ is convex. Compactness of $\tilde{U}(t,x,\alpha)$ follows from the fact that Φ is continuous in $\tilde{u}$.

Remark 1. It should be noted that equations (6) and (7) hold along the trajectories with further details held in [20], [15]. Also, observe that for l, a and b satisfying the assumptions from Section III, the function $\tilde{l}(t,x,\alpha,\tilde{u}) = l(t,x,\Phi(t,x,\alpha,\tilde{u}))$ is convex and Lipschitz in (x,α) and strictly convex in $\tilde{u}$. Consequently, $\tilde{F}_\lambda$ inherits the favorable properties of F_λ.

Remark 2. It is possible that a symmetric obstacle configuration exists such that multiple optimal control/trajectory pairs could exist for a particular example of (5). We note that although the map Φ is affine in $\tilde{u}$, the Valentine transformation that changes (5) into (8)-(9) can allow for the multiple optimal pairs possible in a given example of (5) to be represented by the unique optimal pair for corresponding (8)-(9) via restrictions, projections, etc. (Further details will be included in the upcoming work)

4.2 Weak Solutions to HJB

We now present the main theorem of this paper, which establishes a unique weak solution of the HJB equation in the class of lower semicontinuous functions vanishing at infinity.

Theorem 1. *For the HJB equation derived from (8)-(9), there exists a unique viscosity solution $\hat{V}$ vanishing at infinity. In addition, there exists a unique control $\hat{u}$.*

Proof. In this proof, we establish existence and uniqueness of the control $\hat{u}$ based on the properties of the Hamiltonian $\tilde{H}$ and then establish uniqueness of $\hat{V}$ by showing that (8)-(9) satisfies a set of assumptions and by using a proof in the style of Proposition 2 from the Appendix.

Existence and Uniqueness of $\hat{u}$ Existence and uniqueness of $\hat{u}$ depends on the structure of $\tilde{H}$ and $\tilde{U}(t,x,\alpha)$. First we show continuity and strict convexity of $\tilde{H}$ in $\tilde{u}$. Both of these conditions are connected to the continuity and convexity of Φ in $\tilde{u}$. We note that function Φ is affine in $\tilde{u}$ for all (t,x,α) and thus continuous and convex in $\tilde{u}$. By definitions of $\tilde{F}_\lambda$ and $\tilde{f}$, we know that $\tilde{F}_\lambda$ is strictly convex and $\tilde{f}$ convex in $\tilde{u}$. Both functions are also continuous in $\tilde{u}$. As a result, $\tilde{H}$ is continuous and strictly convex in $\tilde{u}$. Together with Lemma 1, $\tilde{H}$ admits a unique minimizer $\hat{u}$.

Existence and Uniqueness of $\hat{V}$ For existence and uniqueness of $\hat{V}$, we check our problem against the assumptions required in Proposition 2.

A1 We note that both $\tilde{f}$ and $\tilde{F}_\lambda$ are continuous in $(t,\tilde{u})$ for all $x \in \mathbb{X}$, $\alpha \in \mathbb{A}$ and thus Lebesgue-Borel measurable in those arguments. For $\phi \in L^1([0,\infty);\mathbb{R})$, let us choose $\phi(t) = \tilde{l}_- e^{-\lambda t}$, where $\tilde{l}_- = \inf_{(t,x,\alpha,\tilde{u})} |\tilde{l}(t,x,\alpha,\tilde{u})|$.

A2 We need to establish a growth bound for a.e. $t > 0$ and for all $(x,\alpha) \in \mathbb{X} \times \mathbb{A}$, $\tilde{u}(t) \in \tilde{U}(t,x,\alpha)$. First, we rewrite the dynamics for easier computation of bounds.
From (10), we rewrite the dynamic function $\tilde{f}$ as

$$\tilde{f}(t,x,\overrightarrow{\alpha},\tilde{u}) = A(t,x) + B(t,x,\alpha)\tilde{u}(t) \tag{11}$$

where

$$A(t,x) = a(t,x) - b(t,x)b_{K_o}(t,x)^\dagger \nabla \tilde{h(x_s)}^\dagger \nabla \tilde{h(x_s)} a_{K_o}(t,x)$$

and

$$B(t,x,\alpha) = -b(t,x)b_{K_o}(t,x)^\dagger \nabla h(x_s)^\dagger I(\alpha).$$

Going forward, we note that $\tilde{f}$ is bounded in all arguments (a.e. in t); thus, we have

$$|\tilde{f}(t,x,\alpha,\tilde{u})| \leq K_{\tilde{f}}(t)(1 + |(x,\alpha)|). \tag{12}$$

Next,

$$\begin{aligned} |\tilde{F}_\lambda(t,x,\alpha,\tilde{u})| &\leq \sup_{(x,\alpha,\tilde{u})} |\tilde{l}(t,x,\alpha,\tilde{u})| e^{-\lambda t}, \\ &\leq K_{\tilde{F}_\lambda}(t)(1 + |(x,\alpha)|), \end{aligned} \tag{13}$$

where $K_{\tilde{F}_\lambda}(t) := \sup_{(x,\alpha,\tilde{u})} |\tilde{l}(t,x,\alpha,\tilde{u})| e^{-\lambda t}$. Thus, we take $c \in L^1_{loc}([0,\infty);\mathbb{R}^+)$ as

$$c(t) = K_{\tilde{f}}(t) + K_{\tilde{F}_\lambda}(t).$$

A3 Define set-valued maps $\mathscr{F}(t,x,\alpha) := \{\tilde{f}(t,x,\alpha,\tilde{u}) : \tilde{u} \in \tilde{U}(t,x,\alpha)\}$ and $\mathscr{L}(t,x,\alpha) := \{\tilde{F}_\lambda(t,x,\alpha,\tilde{u}) : \tilde{u} \in \tilde{U}(t,x,\alpha)\}$. Since the functions $\tilde{f}$ and $\tilde{F}_\lambda$ are continuous for $(x,\alpha) \in \mathbb{X} \times \mathbb{A}$ and t a.e., one has that the maps $\mathscr{F}$ and $\mathscr{L}$ are continuous. To show the closedness of images, consider the sequence $\{(x_n,\alpha_n)\} \subset \mathbb{X} \times \mathbb{A}$ that converges to a point $(\bar{x},\bar{\alpha}) \in \mathbb{X} \times \mathbb{A}$. It is clear that

$$\begin{aligned} \mathscr{F}(t,x_n,\alpha_n) &\to \mathscr{F}(t,\bar{x},\bar{\alpha}), \\ \mathscr{L}(t,x_n,\alpha_n) &\to \mathscr{L}(t,\bar{x},\bar{\alpha}). \end{aligned}$$

Convexity of the set

$$\{(\tilde{f}(t,x,\alpha,\tilde{u}), \tilde{F}(t,x,\alpha,\tilde{u}) + r) : \tilde{u} \in \tilde{U}(t,x,\alpha), r \geq 0\}$$

in (x,α) follows from the (componentwise) convexity of $\tilde{f}$ (which is derived from the componentwise convexity of f) and $\tilde{F}_\lambda$.

A4 To satisfy the assumption A4 of Proposition 2, we break this item into finding suitable bounds for $\tilde{F}_\lambda$ and $\tilde{f}$. First, fix t and $\tilde{u} \in \tilde{U}(t,x,\alpha)$. Next, for (x,α) and (y,β), we proceed with the bounds on $\tilde{F}_\lambda$. With the assumptions on $\tilde{l}$, we have $|\tilde{F}_\lambda(t,x,\alpha,\tilde{u}) - \tilde{F}_\lambda(t,y,\beta,\tilde{u})| = K_{\tilde{l}}(t)e^{-\lambda t}|(x,\alpha)-(y,\beta)|$.
In order to establish the appropriate bounds on $\tilde{f}$, we consider similar bounds on the matrix-valued functions A and B as given in $A2$. We note that since a and b are Lipschitz and bounded in x for a.e. $t \geq t_0$ and A is independent of α, we can establish Lipschitz bounds for A of the form

$$|A(t,x) - A(t,y)| \leq K_A(t)|(x,\alpha)-(y,\beta)|.$$

For the function B, we have

$$|(B(t,x,\alpha) - B(t,y,\beta))\tilde{u}(t)| \leq K_B(t)\tilde{u}_b|(x,\alpha)-(y,\beta)|,$$

where $\tilde{u}_b = \sup_{\tilde{u}\in\tilde{U}} |\tilde{u}|$. Thus, we conclude

$$|\tilde{f}(t,x,\alpha,\tilde{u}) - \tilde{f}(t,y,\beta,\tilde{u})| \leq C_{\tilde{f}}(t)|(x,\alpha)-(y,\beta)|, \tag{14}$$

where $C_{\tilde{f}}(t) = K_A(t) + K_B(t)\tilde{u}_b$. In other words, A4 is satisfied with

$$k(t) = K_{\tilde{l}}(t)e^{-\lambda t} + C_{\tilde{f}}(t).$$

A5 To show that $k \in \mathscr{L}_{loc}$, we note that the suprema of $K_{\tilde{l}}(t)$, $K_A(t)$, and $K_B(t)$ are finite. Thus we can say $k(t) \leq 3 * \max\{\sup_t K_{\tilde{l}}(t), \sup_t K_A(t), \sup_t K_B(t)\tilde{u}_b\} =: K$. For $J \subset [t_0,\infty)$ and $\sigma \in [t_0,\infty)$, we have

$$\int_J k(\tau)d\tau \leq K\mu(J),$$

$$\sup_{\{J:\mu(J)\leq\sigma\}} \int_J k(\tau)d\tau \leq \sup_{\{J:\mu(J)\leq\sigma\}} K\mu(J). \tag{15}$$

Letting σ approach zero shows that the RHS of (15) approaches zero and thus $k \in \mathscr{L}_{loc}$. Moreover, we get

$$\begin{aligned} \tfrac{1}{t}\textstyle\int_{t_0}^{t}(c(\tau)+k(\tau))d\tau &\leq K\tfrac{t-t_0}{t} + \tfrac{1}{t}\textstyle\int_{t_0}^{t} c(\tau)d\tau \\ &\leq (K+c_{max})(1+\tfrac{t_0}{t}). \end{aligned} \tag{16}$$

As t approaches infinity, we note that the right-hand side of (16) is finite.

A6 We know from A2 that for $t \geq 0$, closed set $\mathbb{X}$, and all $x \in \partial\mathbb{X}$, it follows that $|\tilde{f}(t,x,\alpha,\tilde{u})| + |\tilde{F}_\lambda(t,x,\alpha,\tilde{u})| \leq c(t)(1+\tilde{x}_m)$, where $\tilde{x}_m = \sup\{|(x,\alpha)| : (x,\alpha) \in \partial\mathbb{X}\times\partial\mathbb{A}\}$. Taking the supremum over $\tilde{u} \in \tilde{U}$ yields the assumption A6 with $q(t) = c(t)(1+x_m)$.

A7 By the structure of $\tilde{F}_\lambda$, the assumption A7 readily follows.

A8 The idea of A8 is to show that for a point $(\bar{x},\bar{\alpha}) \in \partial\mathbb{X}\times\partial\mathbb{A}$, there exists a trajectory that lies in the interior of $\mathbb{X}\times\mathbb{A}$ such that $(\bar{x},\bar{\alpha})$ is reached. Prior to problem (8)-(9), we assumed that the dynamics $\dot{x} = a(t,x)+b(t,x)u$ are controllable (which satisfies A8 for (5)). Now we need to show that the controllability property holds also for $\dot{x} = \tilde{f}(t,x,\alpha,\tilde{u})$. Let us note that $\Phi(t,x,\alpha,\cdot)$ is a closed map (in the fourth variable). Since Φ is a closed map in $\tilde{u}$ and the original dynamics are controllable, there exists a control $\tilde{u}$ such that one can reach some final position $x_f \in \partial\mathbb{X}$ with a trajectory in the interior of $\mathbb{X}$ whose dynamics are governed by $\tilde{f}$.

Having established the assumptions A1-A8, it suffices to show that the value function $\hat{V}(t,x,\alpha) = \int_t^\infty \tilde{l}(s,x,\alpha,\hat{u})e^{-\lambda s}ds$ has the following properties: (i) $\hat{V}$ is lower semicontinuous, (ii) Dom $\hat{V}(t,\cdot,\cdot) \neq \emptyset$ for all large $t > t_0$, and (iii)

$$\lim_{t\to\infty} \sup_{(y,\beta)\in \text{Dom } \hat{V}(t,\cdot,\cdot)} |\hat{V}(t,y,\beta)| = 0.$$

We know that $\tilde{l}$ is at least lower semicontinuous in $t,\ x,$ and α and thus (i) is satisfied (e.g., see Remark 1). Similarly, since $\tilde{l}$ is bounded and defined in all necessary arguments, we have (ii). For (iii),

$$\lim_{t\to\infty} \sup_{(y,\beta)\in \text{Dom } \hat{V}(t,\cdot,\cdot)} |\hat{V}(t,y,\beta)| \le \lim_{t\to\infty} \int_t^\infty \tilde{l}_+ e^{-\lambda s} ds,$$

where $\tilde{l}_+ = \sup_{(t,y,\beta)} |\tilde{l}(t,x,\alpha,\hat{u})|$. It is clear that the RHS converges to zero as $t\to\infty$ and thus we establish (iii). From here, we apply Proposition 2 with $(1) \implies (2)$ and have the value function $\hat{V}$ as the unique viscosity solution. □

Remark 3. Note that the classical necessary and sufficient conditions for existence of solutions to (5) *without constraints* can be expressed through PP ([13, pp.102-3]) and HJB ([13, pp.165]) respectively. We stress that existence and uniqueness results are applied to problem (8)-(9) rather than problem (5).

5 SOLVING THE ORIGINAL PROBLEM

5.1 Convergence to Original Infinite-Horizon Problem

Up to this point, we have analyzed (5) by rewritting it as (8)-(9). Now we show how to exploit (5) to solve the infinite-horizon version of (3), that is, we set $t_f = \infty$ and $E(x_f) = 0$ in (3). Recall that it is possible that non-unique trajectory/control pairs $\{(\tilde{x},\tilde{u})\}$ exist that solve infinite-horizon (3). Given (x_λ, u_λ) as a solution of (5), we establish the conditions on (x_λ, u_λ) to be a sufficient approximation for a solution of infinite-horizon (3).

Proposition 1 *Consider the family of cost functions* $\{J_\infty : \mathbb{X}\times U(t)\times[0,\infty)\to\mathbb{R}\}$, *where* $J_\infty(x,u,\lambda) = \int_{t_0}^\infty l(s,x(s),u(s))e^{-\lambda s}ds$ *(λ acts as the indexing variable) with dynamics and constraints from (5) and* $J_\infty(x,u,0) = \int_{t_0}^\infty l(s,x(s),u(s))ds$ *(i.e., the cost* *function for the infinite-horizon variant of (3)). Let* (x_λ, u_λ) *be the associated trajectory/control pair that minimizes* $J_\infty(x,u,\lambda)$ *and* $l(t,x,u)$ *satisfy the conditions posed in Section III. Then* $J_\infty(x,u,\lambda) \xrightarrow{\Gamma} J_\infty(x,u,0)$ *and there exists a trajectory/control pair* (x^*,u^*) *such that* $(x_\lambda,u_\lambda) \to (x^*,u^*)$ *and* (x^*,u^*) *solves the infinite-horizon problem (3).*

Proof. To show $\Gamma-$convergence of $J(x,u,\lambda)$, we consider the standard arguments involving J^{sup} and J^{inf} (related to F^{sup} and F^{inf} from Section II). Since $e^{-\lambda t} \leq 1$ for $\lambda \in [0,\infty)$, we know $l(t,x(t),u(t))e^{-\lambda t} \leq l(t,x(t),u(t))$ and thus $\int_{t_0}^{\infty} l(s,x(s),u(s))e^{-\lambda s} ds \leq \int_{t_0}^{\infty} l(s,x(s),u(s))ds$. From here, it is clear that

$$J(x,u,0) \geq J^{sup}(x,u) := \limsup_{\varepsilon\to 0} J(x_{1,\varepsilon},u_{1,\varepsilon},\varepsilon) \tag{17}$$

for some sequence $(x_{1,\varepsilon}(t),u_{1,\varepsilon}(t)) \to (x(t),u(t))$. To get the necessary inequality

$$J(x,u,0) \leq J^{inf}(x,u) := \liminf_{\varepsilon\to 0} J(x_{2,\varepsilon},u_{2,\varepsilon},\varepsilon), \tag{18}$$

it suffices to define a sequence $(x_{2,\varepsilon}(t),u_{2,\varepsilon}(t))$ such that for a given $(x(t),u(t))$, $(x_{2,\varepsilon}(t),u_{2,\varepsilon}(t))$ converges to $(x(t),u(t))$ and (18) is satisfied.

To establish this sequence, we first consider target (x,u) such that $J(x,u,0) < \infty$. We note that J is continuous in x, u, and ε. Thus, consider $(x_{2,\varepsilon}(t),u_{2,\varepsilon}(t)) := (x(t)+e_1(\varepsilon),u(t)+e_2(\varepsilon))$ such that $J(x_{2,\varepsilon},u_{2,\varepsilon},\varepsilon) \geq J(x,u,0)$ and $(e_1(\varepsilon),e_2(\varepsilon)) \xrightarrow{\varepsilon\to 0} 0$ Here we have $\lim_{\varepsilon\to 0} J(x_{2,\varepsilon},u_{2,\varepsilon},\varepsilon) \geq J(x,u,0)$. Hence, (18) is satisfied and we have shown that $J(x,u,\lambda) \xrightarrow{\Gamma} J(x,u,0)$.

Next, we show that there exists a compact set C with $\inf_{(x,u)} J(x,u,\varepsilon) = \inf_{(x,u)\in C} J(x,u,\varepsilon)$ as $\varepsilon > 0$. We note that, by the properties of the function $l(t,x(t),u(t))$, $J(x,u,\varepsilon)$ is a bounded function. Thus for each $\varepsilon > 0$, we can always find such a set C.

Owing to the previous part of the proof and since $\dot{x} = a(t,x) + b(t,x)u$ is controllable, there exists a pair (x^*,u^*) such that $\int_{t_0}^{\infty} l(s,x^*(s),u^*(s))ds$ is finite and minimal. By Theorem 1, for $\lambda > 0$, there exists a unique optimal trajectory/control pair (x_λ,u_λ) that solves the corresponding optimal control problem (5) with the corresponding λ value.

Since $J(x,u,\lambda) \xrightarrow{\Gamma} J(x,u,0)$ and each pair (x_λ,u_λ) minimizes $J(x,u,\lambda)$ for a given λ, we reach $(x_\lambda,u_\lambda) \to (x^*,u^*)$ [10, Corollary 7.24]. □

Remark 4. One can interpret Proposition 1 as the result of when $\lambda \to 0$ in the sense that we are determining the efficacy of (x_λ,u_λ) when the cost function lacks the exponential decay in time (i.e., $\lambda = 0$).

Remark 5. We take the time to acknowledge the similarity of this approach to that of the "vanishing discount method" of ergodic control theory. In [4] and [5], one requires constructing a Lyapunov function to show precompactness of the family of cost functions $\mathbb{J} = \{J_\infty(\cdot,\cdot,\lambda)\}$. In short, one must rely on the dynamics of the problem (typically a stochastic differential equation, [9]) to obtain convergence for $\mathbb{J}$ and thus $\{(x_\lambda,u_\lambda)\}$. In our results, once Theorem 1 is applied, convergence only depends on the properties of $\mathbb{J}$. With respect to state constrained problems (see [14]), additional assumptions are required on the constraints. In this paper, state constraints are handled via the Valentine transformation and the results of Theorem 1 rather than with Γ-convergence. It should be noted that Γ-convergence can be used independently of any control framework, i.e. the behavior of a family functionals and their respective minimizers.

5.2 Approximation of Original Finite-Time Problem

For many practical problems, finite-horizon formulations are of interest, that is, t_f is finite and $E(\cdot) \not\equiv 0$ in (3). Essentially, we wish to form a relation between the infinite-time horizon problem (5) and the finite-time version of problem (3). Consider the new running cost functional $l'(t,x,u,\lambda)$ defined as

$$l'(t,x,u,\lambda) = \begin{cases} l(t,x,u) \text{ if } t \in [t_0,t_f], \\ \gamma(t,u,\lambda)E(x(t_f),x_f) \text{ if } t > t_f, \end{cases} \tag{19}$$

and thus

$$\begin{aligned} \textstyle\int_{t_0}^{\infty} l'(s,x(s),u(s),\lambda)e^{-\lambda s}ds = &\textstyle\int_{t_0}^{t_f} l(s,x(s),u(s))e^{-\lambda s}ds \\ &+ \textstyle\int_{t_f}^{\infty} \gamma(s,u,\lambda)E(x(t_f),x_f)e^{-\lambda s}ds. \end{aligned} \tag{20}$$

We choose $\gamma(t,u,\lambda)$ such that Theorem 1 holds and Proposition 1 is satisfied with the necessary modifications (e.g., restriction of one's interest to the interval $[t_0,t_f]$). In general the function α is not unique, so its choice is to personal preference. In the example section, we select $\gamma(t,u,\lambda)$ equal to $\lambda e^{\lambda t_f}(u^T R_s u + 1)$ with a user-selected positive definite matrix R_s.

In order to assess the proposed approximation numerically, we consider the function

$$E_D(\lambda,t_f) := \frac{|J_\infty(t_0,\hat{x},\hat{u},\lambda) - J(t_0,t_f,\hat{x}_f,\hat{x},\hat{u})|}{J(t_0,t_f,\hat{x}_f,\hat{x},\hat{u})}, \tag{21}$$

with $(\hat{x},\hat{u})$ as the trajectory/control pair which minimizes J_∞. The function E_D acts as a (relative) error function between the given cost function J and the infinite-horizon approximation J_∞.

6 EXAMPLE

This section presents the numerical example in which we are solving a path planing problem in 3D with obstacles for the nonlinear dynamics which represents the AUV. We present a test case for a finite-time application and continue with the calculation of E_D for multiple values of λ.

6.1 AUV

According to the more common notation of rigid-body models found in literature, this example uses ω for the state x in (3) and τ corresponds to u. Consider the problem

$$\min_{(\omega,\tau)} \ J(t_0,t_f,\omega,\tau) = \textstyle\int_{t_0}^{t_f} \omega(s)^T Q\omega(s) + \tau(s)^T R\tau(s)ds \tag{22}$$

subject to

$$\begin{aligned}\dot{\omega}_s &= m^{-1}(\hat{\omega}_s(\omega_{ns}) - D_s\omega_s + \tau_s),\\ \dot{\omega}_{ns} &= I_{CG}^{-1}((\widehat{I_{CG}\omega_{ns}})(\omega_{ns}) - D_r\omega_{ns} + \tau_r),\\ (\omega(t_0), \omega(t_f)) &= (\omega_0, \omega_f),\end{aligned} \tag{23}$$

with $\omega = [\omega_s; \omega_{ns}]$ and $\tau = [\tau_s; \tau_r]$ where $\omega_s = [x; y; z] \in \mathbb{R}^3$ are the translation states, $\omega_{ns} = [\rho; \theta; \psi] \in [-\pi, \pi]^3$ are the rotational states and controls $\tau_s,\ \tau_r \in \mathbb{R}^3$. We define m as the mass of the AUV robot, 3×3 matrices $D_s,\ D_r$ are damping matrices, and I_{CG} is the inertia matrix which is symmetric and positive definite of the form

$$I_{CG} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}.$$

We also include constraints

$$\begin{aligned}(\tfrac{x-8}{1.7})^2 + (\tfrac{y-8}{1.7})^2 + (\tfrac{z-8}{1.7})^2 &\geq 1,\\ (\tfrac{x-1}{1.7})^2 + (\tfrac{y-1}{1.7})^2 + (\tfrac{z-2}{1.7})^2 &\geq 1,\\ (\tfrac{x-4.5}{2.7})^2 + (\tfrac{y-5}{2.7})^2 + (\tfrac{z-8}{2.7})^2 &\geq 1,\\ (\tfrac{x-5.5}{2.7})^2 + (\tfrac{y-5}{2.7})^2 + (\tfrac{z-2}{2.7})^2 &\geq 1.\end{aligned} \tag{24}$$

We define constraints (24) in the form $I^{-1}(h_L)h(x_s) \geq I^{-1}(h_L)h_L$ for ease of numerical computation. Furthermore, $h_L = [1.7^2; 1.7^2; 2.7^2; 2.7^2]$ is understood with $d_i = 0.7$ (0.5 accounts for the radius of BlueROV, 0.2 gives additional buffing from obstacles; all values are given in meters). Define spaces $\mathbb{X} = [0, 10]^3 \times [-\pi, \pi]^3$ and $\mathbb{U}(t) = [-50, 50]^6$. We set values $Q = \text{diag}\{1; 1; 1; 50; 50; 50\}$, $R = \text{diag}\{1; 10; 10; 1; 5; 10\}$, $R_s = 0.02 I_6$, $I_x = I_y = 3.45$, $I_{xz} = I_{zx} = -1.28 \times 10^{-15}$, $I_z = 2.2$, $I_{xy} = I_{yx} = I_{yz} = I_{zy} = 0$, $D_s = \text{diag}\{0.1; 5; 5\}$, $D_r = \text{diag}\{5; 5; 0.1\}$, $\omega_0 = [10; 10; 10; 0; 0; 0]$, and $\omega_f = \mathbf{0}_{6\times 1}$.

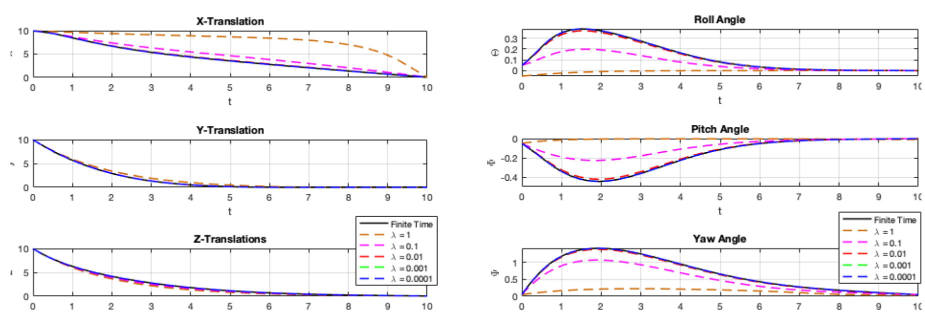

Fig. 1: a) Translations of AUV, b) Angles of AUV

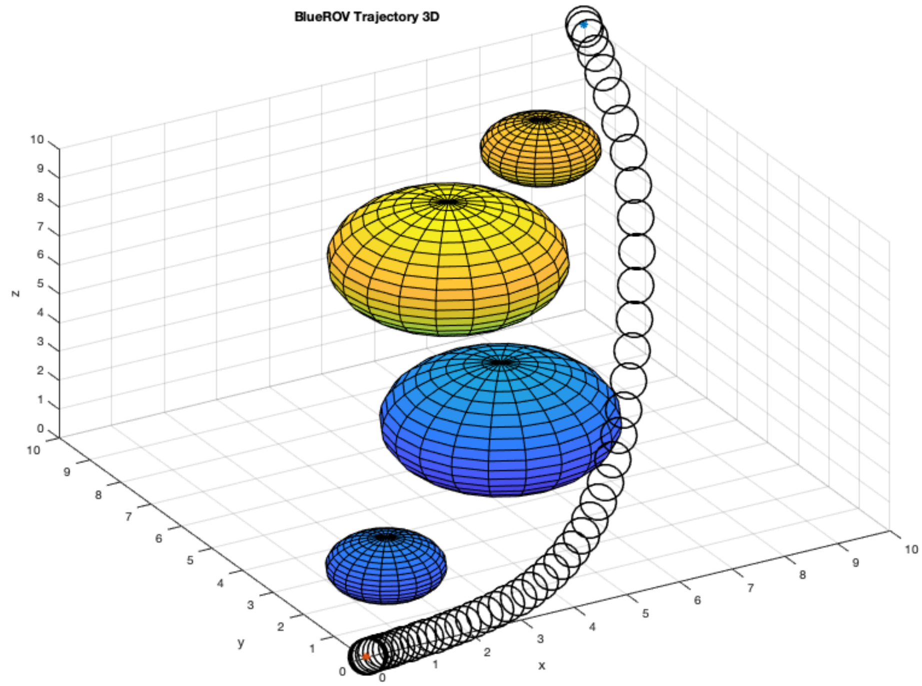

Fig. 2: Trajectory of AUV

For finite-time problem approximation, we choose $t_f = 10$ and also take into consideration the number of algorithm nodes (a_n) used for calculation with `DIDO`. In `DIDO`, a_n relates to the discretization of the time interval $[t_0, t_f]$ for computation of cost function $J(t_0, t_f, \omega, \tau)$.

	$a_n = 50$		$a_n = 20$	
λ	$E_D(\lambda, t_f)$	runtime [s]	$E_D(\lambda, t_f)$	runtime [s]
1	0.84	185.44	0.83	58.34
0.1	0.26	149.72	0.25	37.84
0.01	0.03	148.23	2.73e-02	35.81
0.001	3e-03	141.58	2.8e-03	42.80
0.0001	2.56e-04	143.55	2.99e-04	45.67

Table 1: Relations between λ, $E_D(\lambda, t_f)$, and computational runtime with $a_n = 50$ and $a_n = 20$.

Simulation results for the path planing problem in 3D with 3 obstacles for a AUV are shown in figures 1 and 2. As we observe in Fig.1, as λ approaches 0, both translations and rotations converge to the resulting finite-time behavior. This behavior is also reflected in Table 1 with respect to E_D for both $a_n = 20$ and $a_n = 50$.

7 CONCLUSIONS AND FUTURE WORK

We have established uniqueness and existence results for a class of optimal control problem with constraints employing results for the HJB equation and Pontryagin's Principle. We have shown how Γ-convergence of cost functions can be used to solve constrained finite- and infinite-horizon optimal problems. Our framework is illustrated through solving a problem of 3D path planing with obstacles for the nonlinear dynamics of an AUV.

The principal research avenue is to investigate stability and robustness of the solutions obtained via the proposed framework. In other words, we do not want to *a priori* assume that the trajectories live in a bounded set, but the trajectory boundedness for initial conditions of interest ought to be obtained via the proposed optimization. Also, we plan to extend current results to path planning problems with dynamics corresponding to aerial and marine robots as well as the coverage problem for mobile agents with sensors such as cameras, sonars or lidars. We may also consider dynamic obstacles (e.g., mobile agents that intersect with floating debris or local wildlife).

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APPENDIX

In order to make this paper as self-contained as possible and for the reader's convenience, we establish some additional notation needed for this section. Let $B_r(x)$ be the closed ball in $\mathbb{R}^k$ of radius $r > 0$ centered at $x \in \mathbb{R}^k$. For a nonempty set $C \subset \mathbb{R}^k$, we denote its convex hull by $co(C)$.The negative polar cone of C is defined as $C^- := \{p \in \mathbb{R}^k : \langle p, c\rangle \leq 0 \text{ for all } c \in C\}$. Denote the set distance from $x \in \mathbb{R}^k$ to C by $d_C(x) := \inf\{|x-y| : \ y \in C\}$.

For a nonempty set $C \subset \mathbb{R}^k$, collection of nonempty subsets $\{C_w\}_{w \in C_2}$, and $w_0 \in C_2$, let

$$\operatorname{Lim}\sup_{w \to w_0} C_w = \{v \in \mathbb{R}^k : \liminf_{w \to w_0} d_{C_w}(v) = 0\}$$

and

$$\operatorname{Lim}\inf_{w\to w_0} C_w = \{v\in\mathbb{R}^k:\ \limsup_{w\to w_0} d_{C_w}(v)=0\}.$$

For $x\in\overline{C}$ define $T_C(x):=\{v\in\mathbb{R}^k : \liminf_{h\to 0^+} d_A(x+hv)/h=0\}$ and $N_C(x):=\operatorname{Lim}\sup_{y\to_C x} T_C(y)^-$.
Finally, denote the epigraph of a function φ as epi φ.

Below we state the necessary assumptions and proposition required for the proof of Theorem 1.

A1 For all $x\in\mathbb{R}^n$, the mappings $\tilde f(\cdot,x,\cdot)$ and $\tilde F(\cdot,x,\cdot)$ are Lebesgue-Borel measurable and there exists $\phi\in L^1([0,\infty);\mathbb{R})$ such that $\tilde F(t,x,u)\ge\phi(t)$ for a.e. $t\ge 0$ and all $(x,u)\in\mathbb{R}^n\times\mathbb{R}^m$.

A2 There exists a function $c\in L^1_{loc}([0,\infty);\mathbb{R}^+)$ such that for a.e. $t\ge 0$ and for all $x\in\mathbb{R}^n$, $u\in U(t)$

$$|\tilde f(t,x,u)|+|\tilde F(t,x,u)|\le c(t)(1+|x|).$$

A3 For a.e. $t\ge 0$ and all $x\in\mathbb{R}^n$, the set-valued map

$$y\to\{(\tilde f(t,y,u),\tilde F(t,y,u)):u\in U(t)\}$$

is continuous with closed images, and the set

$$\{(\tilde f(t,x,u),\tilde F(t,x,u)+r):u\in U(t),\ r\ge 0\}$$

is convex.

A4 There exists a function $k\in L^1_{loc}([0,\infty);\mathbb{R}^+)$ such that for a.e. $t\ge 0$ and for all $x,y\in\mathbb{R}^n$, $u\in U(t)$: $|\tilde f(t,x,u)-\tilde f(t,y,u)|+|\tilde F(t,x,u)-\tilde F(t,y,u)|\le k(t)|x-y|$.

A5 $k\in\mathscr{L}_{loc}$ and $\limsup_{t\to\infty}\frac{1}{t}\int_0^t(c(s)+k(s))ds<\infty$.

A6 There exists a function $q\in\mathscr{L}_{loc}$ such that for a.e. $t\ge 0$

$$\sup_{u\in U(t)}(|\tilde f(t,x,u)|+|\tilde F(t,x,u)|)\le q(t),\ \text{for all } x\in\partial C$$

where $C\subset\mathbb{R}^n$ is closed.

A7 Let $V:[0,\infty)\times C\to\mathbb{R}\cup\{\pm\infty\}$ be the value function of (8)-(9). Then assume Dom $V\ne\varnothing$ and there exist $T>0$ and $\psi\in L^1([T,\infty);\mathbb{R}^+)$ such that for all $(t_0,x_0)\in$ Dom $V\cap([T,\infty)\times\mathbb{R}^n)$ and any feasible trajectory-control pair $(x(\cdot),u(\cdot))$ on $[t_0,\infty)$, with $x(t_0)=x_0$,

$$|\tilde F(t,x(t),u(t))|\le\psi(t)\ \text{for a.e. } t\ge t_0.$$

A8 There exist $\eta,\ r>0$ and $M\ge 0$ such that for a.e. $t>0$ and any $y\in\partial C+\eta B_1(\mathbf{0})$, and any $v\in\tilde f(t,y,U(t))$, with $\inf_{n\in N^1_{y,\eta}}\langle n,v\rangle\le 0$, we can find $w\in\tilde f(t,y,U(t))\cap B_M(v)$ satisfying

$$\inf_{n\in N^1_{y,\eta}}\{\langle n,w\rangle,\ \langle n,w-v\rangle\}\ge r,$$

where $N^1_{y,\eta}:=\{n\in\partial B_1(\mathbf{0}):n\in\overline{co}N_C(x),\ x\in\partial C\cap B_\eta(y)\}$.

Proposition 2 (Theorem 3.3, [2]) *Let $W : [0,\infty) \times C \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function such that Dom $V(t,\cdot) \subset$ Dom $W(t,\cdot) \neq \emptyset$ for all large $t > 0$ and*

$$\lim_{t\to\infty} \sup_{y\in Dom\, W(t,\cdot)} |W(t,y)| = 0. \tag{25}$$

Then the following statements are equivalent:

1. *W = V;*
2. *W is a weak (or viscosity) solution of HJB equation on $(0,\infty) \times C$ and $t \to$ epi $W(t,\cdot)$ is locally absolutely continuous.*

In addition, V is the unique weak solution satisfying (25) with locally absolutely continuous $t \to$ epi $V(t,\cdot)$.