Numerically Efficient Agents-to-Group H_∞ Analysis

Ivica Nakić *, **Domagoj Tolić** **, **Ivana Palunko** ***, **Zoran Tomljanović** ****

* *Department of Mathematics, University of Zagreb, Croatia (e-mail:nakic@math.hr)*
** *LARIAT – Laboratory for Intelligent Autonomous Systems & RIT Croatia, Dubrovnik, Croatia (e-mail: domagoj.tolic@croatia.rit.edu)*
*** *LARIAT – Laboratory for Intelligent Autonomous Systems, University of Dubrovnik, Croatia (e-mail: ivana.palunko@unidu.hr)*
**** *Department of Mathematics, J. J. Strossmayer University of Osijek, Croatia (e-mail: ztomljan@mathos.hr)*

Abstract: This paper proposes a numerically efficient approach for computing the maximal/minimal impact a subset of agents has on the cooperative system. For instance, if one is able to disturb/bolster several agents so as to maximally disturb/bolster the entire team, which agents to choose and what kind of inputs to apply? We quantify the agents-to-team impacts in terms of H_∞ norm whereas output synchronization is taken as the underlying cooperative control scheme. Sufficient conditions on agents' parameters, synchronization gains and topology are provided such that the associated H_∞ norm attains its maximum for constant agents' disturbances. Linear second-order agent dynamics and weighted undirected topologies are considered. Our analyses also provide directions towards improving graph design and tuning/selecting cooperative control mechanisms. Lastly, numerical examples, some of which include forty thousand agents, are provided.

Keywords: multi-agent systems, H_∞ norm, network robustness/resiliency, output synchronization, linear systems

1. INTRODUCTION

Network topology and agent dynamics play crucial roles in Multi-Agent System (MAS) stability and performance as discussed in Olfati-Saber and Murray (2004); Ren and Beard (2008); Tolić et al. (2015a). Thus, we tackle the following question: *Given some topology and agent dynamics, if one is to disturb (or bolster) one or more agents in order to undermine (or enhance) efforts of the entire team, which agent(s) and disturbances to choose?* Subsequently, the system designer might want to modify the graph edge weights, remove or add communication/sensing links in an effort to preclude unfavorable cooperative behaviors. Similar analyses are routinely carried out for vibrational systems (e.g., buildings, bridges, etc.), which can be modeled as MASs Tomljanović and Voigt (2020); Nakić et al. (2019); Veselić (2011). Related ideas are also found in social networks, economics, political and health care systems (e.g., sociometric stars, invisible colleges, outsiders or cliques Leinhardt (1977); Zhao et al. (2009); Proskurnikov and Tempo (2017); Peng et al. (2018)) as well as in smart grids Sorrentino et al. (2013); Dileep (2020).

In this work, we investigate H_∞ norms of MASs when disturbances are applied to a subset of agents whilst the group behavior is of interest. We allow for scenarios in which some exogenous signals simultaneously disturb more than one agent while the impact of this signal on each agent may be different (captured by the agents' disturbance weights as explained below). MASs are modeled as undirected weighted connected graphs with nodes/agents being linear second order Ordinary Differential Equations (ODEs) and aiming for decentralized output synchronization. As we are interested in MAS stability/performance with respect to the equilibrium manifold, whose existence stems from the output synchronization coordination scheme, we reduce the whole MAS system by factoring out the eigenvector corresponding to the zero eigenvalue of the graph Laplacian. As the computational burden of calculating H_∞ norm is significant, precluding direct calculations in larger MASs, we further analyze the corresponding transfer functions. As a result, we prove that the H_∞ norm is attained in zero for a large class of MASs, thus obtaining a simple formula for the H_∞ norm that allows one to treat MASs with a great number of agents. The present work is a continuation of Nakić et al. (2021), where the impact of merely one agent on the team is considered.

As already mentioned, computational costs behind finding H_∞ norms are at times quite prohibitive even for systems of moderate sizes, not to mention when various system

⋆ This work was supported in part by Croatian Science Foundation under the projects IP-2016-06-2468 (ConDyS), IP-2019-04-6774 (VIMS) and in part by the European Commission under project SeaClear, grant 871295.

parameters or input-output pairs need to be considered. Therefore, efficient calculations of the H_∞ norm are intensively investigated, especially when a large number of states (that is, agents in our case) is encountered Zhou et al. (1996); Boyd and Balakrishnan (1990); Aliyev et al. (2017); Chahlaoui et al. (2004); Benner et al. (2012); Guglielmi et al. (2013); Mitchell and Overton (2016).

Works relating the H_∞ norm and MASs typically focus on syntheses Saboori and Khorasani (2014); Elahi et al. (2019); Yaghmaie et al. (2019); Stoorvogel et al. (2019); Li et al. (2009) whilst our paper is primarily concerned with analyses. For example, Saboori and Khorasani (2014); Elahi et al. (2019) boil down to Linear Matrix Inequalities (LMIs) whereas Yaghmaie et al. (2019) builds upon game theory and dynamic programming to provide sufficient conditions for controller design yielding MAS synchronization with a prescribed $\mathcal{L}_2$-gain. The authors in Stoorvogel et al. (2019) provide sufficient and necessary conditions for decentralized H_∞ and H_2 control design over directed graphs employing the algebraic Riccati equation (ARE) or direct eigenstructure assignment. Even though it also focuses on syntheses, the most similar work to ours is Li et al. (2009) as it performs system reduction to mitigate the H_∞-related computational burden and searches for performance bottlenecks on the individual agent level. Therefore, unlike in the present work, the team performance improvement guidelines in Li et al. (2009) boil down merely to individual agent modifications via pinning control (i.e., via adding self-loops) irrespective of the topology and synchronization gains. In addition, Li et al. (2009) considers merely stability of the origin (not of the equilibrium manifold as done herein) and does not tackle H_∞ norm computations. The H_2 norm as the MAS performance measure will be treated in a subsequent publication.

The principal contributions of this paper are twofold: a) sufficient conditions for attaining the H_∞ norm in zero, which significantly reduces the entailed computational costs, for a large class of MASs and exogeneous disturbances; and b) a finding that, in order to maximize the excitation of the entire system, it is sufficient to have a single exogenous signal applied to all agents with the corresponding weights given by the Fiedler vector of graph Laplacian matrix. Since many MASs are designed to achieve asymptotic (i.e., steady-state) goals (e.g., output synchronization), it is not surprising they behave like low-pass filters so that the modulus of associated transfer functions attain their maxima at the zero frequency corresponding to the H_∞ norms.

The remainder of the paper is organized as follows. Section 2 introduces the notation and basic definitions. In Section 3 we formulate the agents-to-group H_∞ robustness problem and propose a methodology to solve it in Section 4. Section 5 provides numerical examples whereas the conclusions and future work are in Section 6.

2. PRELIMINARIES

2.1 Notation

We often use the shorthand notation $(x, y) := [x^\top \quad y^\top]^\top$. The dimension of a vector x is n_x whereas $\|\cdot\|$ denotes the Euclidean norm of a vector. If the argument of $\|\cdot\|$ is a matrix, then it denotes the induced matrix 2-norm. The $n\times m$ matrix with all zero entries is $\mathbf{0}_{n\times m}$. The $n\times n$ identity matrix is I_n. For brevity, we use "w.r.t." instead of "with respect to".

2.2 Graph Theory

An *undirected weighted graph* is given by a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \{w_{jk}\}_{j,k=1}^n)$, where $\mathcal{V} = \{v_1, \dots, v_n\}$ is a nonempty set of *nodes*, $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of *edges* and $w_{jk} \geq 0$ are edge weights such that $w_{jk} = w_{kj}$ for all j, k and $w_{jk} > 0$ if and only if $(j, k) \in \mathcal{E}$. When the edge (i, j), $i \neq j$, belongs to $\mathcal{E}$, it means that there are information flows from the node v_i to the node v_j. The set of *neighbors* of the node v_i is $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. The corresponding graph Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as

$$L = [l_{ij}], \quad l_{ij} = \begin{cases} -w_{ij}, & j \in \mathcal{N}_i, \\ \sum_{k\in\mathcal{N}_i} w_{ik}, & j = i, \\ 0, & \text{otherwise}, \end{cases}$$

making it symmetric and positive semi-definite.

2.3 H_∞ norm

For functions F in the space

$$\mathcal{H}_\infty = \left\{ F : \mathbb{C}^+ \to \mathbb{C}^{m\times\ell} \;\middle|\; F \text{ analytic s.t. } \sup_{\lambda\in\mathbb{C}^+} \overline{\sigma}(F(\lambda)) < \infty \right\},$$

where $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$ and $\overline{\sigma}(\cdot)$ is the largest singular value of a matrix, the H_∞ norm is defined as (Zhou et al., 1996, Chap. 3)

$$\|F\|_\infty = \sup_{\lambda\in\mathbb{C}^+} \overline{\sigma}(F(\lambda)) = \sup_{\omega\in\mathbb{R}} \overline{\sigma}(F(\mathrm{i}\omega)).$$

3. PROBLEM STATEMENT

Consider n linear agents given by

$$\ddot{\chi}_i = -T_s\dot{\chi}_i + K_s\upsilon_i + \eta_i\omega_i, \qquad T_s, K_s > 0, \tag{1}$$

where $\chi_i \in \mathbb{R}$ is the state, $\upsilon_i \in \mathbb{R}$ is the input, $\omega_i \in \mathbb{R}$ is the exogenous disturbance of the i^{th} agent, $i \in \{1, \dots, n\}$, and $\eta_i \in \mathbb{R}$ is the corresponding disturbance weight.

The agent dynamics (1) represent a realistic double integrator. These dynamics allow for more specific results in the upcoming sections while still being general enough owing to the following well-known fact: for fully actuated mechanical systems a simple change of control variable transforms their dynamics into a double integrator Ren and Beard (2008). Similarly, many systems with low-level controllers can be satisfactorily approximated with second order dynamics.

A widely utilized decentralized output-feedback policy to achieve network synchronization Ren and Beard (2008) is

$$\upsilon_i = -K\hat{C}\sum_{j\in\mathcal{N}_i} w_{ij}\left(\begin{bmatrix}\chi_i \\ \dot{\chi}_i\end{bmatrix} - \begin{bmatrix}\chi_j \\ \dot{\chi}_j\end{bmatrix} \right), \tag{2}$$

where $K > 0$ and $\hat{C} = [c_1 \quad c_2]$ with $c_1, c_2 > 0$.

A standing assumption herein is that the underlying topology/graph is connected. If it is not connected, the corresponding MAS can be split into m MASs, with $m > 1$ representing the number of connected components, that can be analyzed independently via the methodology presented below. In addition, we suppose the topology is known. If not, one can readily employ topology discovery Bao and Garcia-Luna-Aceves (1999); Dimakopoulos and Pitoura (2003).

As we also want to treat scenarios in which the disturbances ω_i need not all to be independent, let us suppose that among the exogenous disturbances $\{\omega_1,\dots,\omega_n\}$ there are $k \in \{1,\dots,n\}$ different ones $\{\omega_{i_1},\dots,\omega_{i_k}\}$ (e.g., the same ocean waves or wind gusts concurrently disturb several agents). Then $[\omega_1\ \dots\ \omega_n]^\top = H\,[\omega_{i_1}\ \dots\ \omega_{i_k}]^\top$, where the matrix $H \in \mathbb{R}^{n\times k}$ is given by

$$H_{jl} = \begin{cases} 1, & \omega_j = \omega_{i_l} \\ 0, & \text{otherwise} \end{cases}, \quad j = 1,\dots,n,\ \ l = 1,\dots,k.$$

With $E = \operatorname{diag}(\eta_1,\dots,\eta_n)H \in \mathbb{R}^{n\times k}$ we denote the corresponding disturbance matrix. We assume that $\eta_i \neq 0$ if $H_{ii} \neq 0$, i.e., if the i^{th} agent is disturbed, than the corresponding weight is not zero. This in particular implies that E is a full rank matrix.

Utilizing the Laplacian matrix L of the underlying topology, the closed-loop dynamics become

$$\ddot{\chi} + \big(\underbrace{T_s}_{:=\beta} I_N + L\underbrace{K_s K c_2}_{:=\alpha}\big)\dot{\chi} + L\underbrace{K_s K c_1}_{:=\gamma}\chi = E\omega, \tag{3}$$

where $\chi := (\chi_1,\dots,\chi_n)$ and $\omega := (\omega_{i_1},\dots,\omega_{i_k})$.

The control law (2) seeks for agreement/consensus, regardless of where that agreement is obtained. Consequently, the closed-loop system (3) is characterized with the equilibrium manifold $\chi_1 = \dots = \chi_n, \dot{\chi}_1 = \dots = \dot{\chi}_n = 0$, rather than with a sole equilibrium point (in the origin, for instance). Hence, attention needs to be paid to the existence of the consensus manifold as shown in Tolić (2013). In a nutshell, the (output) norms need to be taken w.r.t. sets, rather than w.r.t. points as is typically done. Throughout this paper, the w.r.t set notion is noticeable whenever dealing with the zero eigenvalue of L and associated eigenspace.

We are now ready to state the main problem solved herein.

Problem 1. Efficiently calculate H_∞ norm of system (3) from ω to χ w.r.t. consensus manifold. In addition, analyze the impact various disturbance matrices E and their entries $\eta_{i_1},\dots,\eta_{i_k}$ have on this H_∞ norm.

4. METHODOLOGY

The first step in our approach is closed-loop dynamics reduction in order to attain the transfer function formula. Problem 1, not yet taking into account that we are interested in motion only w.r.t. consensus manifold, boils down to the following linear time-invariant system

$$\dot{x} = Ax + B\omega,$$
$$y = Cx,$$

where $x := (\chi,\dot{\chi})$ while the system matrices $A \in \mathbb{R}^{2n\times 2n}$, $B \in \mathbb{R}^{2n\times k}$ and $C \in \mathbb{R}^{n\times 2n}$ are given by

$$A = \begin{bmatrix} \mathbf{0}_{n\times n} & I_n \\ -\gamma L & -\beta I_n - \alpha L \end{bmatrix}, \quad \alpha,\beta,\gamma > 0,$$
$$B = \begin{bmatrix} \mathbf{0}_{n\times k} \\ E \end{bmatrix}, \quad C = [I_n\ \ \mathbf{0}_{n\times n}]. \tag{4}$$

A detailed derivation of the above expressions (with a different matrix B, but the difference in derivation is trivial) is found in Nakić et al. (2021).

From the construction of L and the connectivity assumption, one knows that the algebraic multiplicity of its zero eigenvalue is one. As discussed in Olfati-Saber and Murray (2004) and Ren and Beard (2008), the corresponding eigenvector is $[1,1,\dots,1]^\top$ and the subspace spanned by this vector is the equilibrium manifold. To obtain the system which describes the motion w.r.t. consensus manifold, we perform the reduction delineated in Nakić et al. (2021) obtaining the following system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\omega$$
$$y = \tilde{C}\tilde{x},$$

where

$$\tilde{A} = \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & V^\top \\ -\gamma L V & -\beta I_n - \alpha L \end{bmatrix},$$
$$\tilde{B} = \begin{bmatrix} \mathbf{0}_{(n-1)\times k} \\ E \end{bmatrix}, \quad \tilde{C} = [V\ \ \mathbf{0}_{n\times n}].$$

The columns of matrix $V \in \mathbb{R}^{n\times(n-1)}$ span the subspace $\{\mathbf{1}\}^\perp$, where $\mathbf{1} = [1\ \dots\ 1]^\top$ and V satisfies $V^\top V = I$. Here, $\tilde{x} \in \mathbb{R}^{n-1}$ and can be thought of as elements from the quotient space $\mathbb{R}^n$ by the equilibrium manifold. Repeating exactly the same calculations as in Nakić et al. (2021), but this time with the matrix E instead of the i^{th} canonical vector e_i found in Nakić et al. (2021), we obtain the following expressions for the transfer function $F(s) = \tilde{C}(\mathrm{i}s - \tilde{A})^{-1}\tilde{B}$ of interest:

$$F(0) = \frac{1}{\gamma} L^+ E, \tag{5}$$

where L^+ denotes the Moore-Penrose pseudoinverse of L, and for $s \neq 0$:

$$F(s) = \frac{1}{\gamma + \mathrm{i}s\alpha} V V^T (L - \mu(s) I_n)^{-1} E, \tag{6}$$

where

$$\mu(s) = \frac{s^2 - \mathrm{i}s\beta}{\gamma + \mathrm{i}s\alpha}.$$

4.1 Main Results

It is well-known that the H_∞ property inferred in the following theorem holds for positive systems Rantzer (2011), but the systems we are interesting in are not necessarily positive.

Theorem 1. Suppose that $\gamma \le \alpha\beta$ or that $\gamma > \alpha\beta$ and $\|L\| \le \frac{\beta^2}{2(\gamma-\alpha\beta)}$. Then we have

$$\|F\|_\infty = \overline{\sigma}(F(0)).$$

Proof: First note that $\overline{\sigma}(F(s))^2 = \|F(s)F(s)^*\|$. Using (6), we obtain

$$\overline{\sigma}(F(s))^2 \cdot (\gamma^2 + s^2\alpha^2) =$$
$$= \|VV^\top (L - \mu(s)I_n)^{-1} E E^\top (L - \overline{\mu(s)} I_n)^{-1} V V^\top\|.$$

Using the well-known formula $LL^+ = L^+L = I_n - 1/n\mathbf{1}\mathbf{1}^\top$, we reach $(L - \mu(s)I_n)^{-1} = (I_n - 1/n\mathbf{1}\mathbf{1}^\top - \mu(s)L^+)^{-1}L^+$. Taking into account that $VV^\top = I - 1/n\mathbf{1}\mathbf{1}^\top$, as $VV^\top$ is the orthogonal projection to the orthogonal complement of the subspace spanned by $\mathbf{1}$, we have
$VV^\top(I_n - 1/n\mathbf{1}\mathbf{1}^\top - \mu(s)L^+)^{-1} = VV^\top(I_n - \mu(s)L^+)^{-1}$.

Taking the above and (5) into consideration, we obtain

$$\begin{aligned}\overline{\sigma}(F(s))^2 \cdot (1+s^2\alpha^2\gamma^{-2}) &= \qquad (7)\\ &= \|VV^\top V(s)F(0)F(0)^*V(s)^*VV^\top\| \\ &\leq \|V(s)F(0)F(0)^*V(s)^*\|,\end{aligned}$$

where $V(s) = (I_n - \mu(s)L^+)^{-1} \in \mathbb{C}^{n\times n}$.

Next, we calculate

$$\begin{aligned}V(s)V(s)^* &= (I-\mu(s)L^+)^{-1}(I-\overline{\mu(s)}L^+)^{-1}\\ &= \Big((I-\overline{\mu(s)}L^+)(I-\mu(s)L^+)\Big)^{-1}.\end{aligned}$$

Our goal is to show that under the assumptions of the theorem, we have $\|V(s)V(s)^*\| \leq 1$. Obviously, using the formula above, it is sufficient to show $((I-\overline{\mu(s)}L^+)(I-\mu(s)L^+)x, x) \geq (x,x)$ for all $x \in \mathbb{R}^n$. By expanding the last relation, one obtains

$$|\mu(s)|^2\|L^+x\|^2 - 2\Re\mu(s)(L^+x,x) \geq 0.$$

Since $\Re\mu(s) = s^2(\gamma-\alpha\beta)/(\gamma^2+s^2\alpha^2)$, if we have $\gamma \leq \alpha\beta$, the last inequality is obviously satisfied. If, on the other hand we have $\gamma > \alpha\beta$ and $\|L\| \leq \frac{\beta^2}{2(\gamma-\alpha\beta)}$, then

$$\begin{aligned}&|\mu(s)|^2\|L^+x\|^2 - 2\Re\mu(s)(L^+x,x)\\ &\geq \|L^+x\|(|\mu(s)|^2\|L^+x\| - 2\Re\mu(s)\|x\|).\end{aligned}$$

As $L^+\mathbf{1} = 0$, it is sufficient to restrict our considerations to those x that are orthogonal to $\mathbf{1}$. For such x we have $\|x\| = \|LL^+x\| \leq \|L\|\|L^+x\|$; hence, $\|L^+x\| \geq \|L\|^{-1}\|x\|$. Using $|\mu(s)| = s^2(s^2+\beta^2)/(\gamma^2+s^2\alpha^2)$, one reaches

$$\begin{aligned}&|\mu(s)|^2\|L^+x\|^2 - 2\Re\mu(s)(L^+x,x)\\ &\geq \frac{s^2}{\gamma^2+s^2\alpha^2}\|L^+x\|\|x\|\left((s^2+\beta^2)\|L\|^{-1} - 2(\gamma-\alpha\beta)\right)\\ &\geq \frac{s^2}{\gamma^2+s^2\alpha^2}\|L^+x\|\|x\|\left(\beta^2\|L\|^{-1} - 2(\gamma-\alpha\beta)\right)\\ &\geq \frac{s^2}{\gamma^2+s^2\alpha^2}\|L^+x\|\|x\|\left(\beta^2\frac{2(\gamma-\alpha\beta)}{\beta^2} - 2(\gamma-\alpha\beta)\right) \geq 0.\end{aligned}$$

We now apply (Horn and Johnson, 2012, Corollary 4.5.11) with $S = V(s)$ and $A = F(0)F(0)^*$ and obtain that for all $s > 0$ there exist numbers $\theta(s)$ such that $0 \leq \theta(s) \leq 1$ such that $\lambda_n(V(s)F(0)F(0)^*V(s)^*) = \theta(s)\lambda_n(F(0)F(0))$, where $\lambda_n(M)$ denotes the largest eigenvalue of the matrix M. This can be written as

$\|V(s)F(0)F(0)^*V(s)^*\| = \theta(s)\|F(0)F(0)^*\| \leq \|F(0)F(0)^*\|$.

Now (7) implies

$$\overline{\sigma}(F(s))^2 \leq \frac{\gamma^2}{\gamma^2+s^2\alpha^2}\overline{\sigma}(F(0))^2.$$

Since $0 < \frac{\gamma^2}{\gamma^2+s^2\alpha^2} < 1$ for all $s > 0$, the theorem statement follows. □

Corollary 1. Suppose that $\gamma \leq \alpha\beta$ or that $\gamma > \alpha\beta$ and $\|L\| \leq \frac{\beta^2}{2(\gamma-\alpha\beta)}$. Then (5) gives

$$\|F\|_\infty = \frac{1}{\gamma}\overline{\sigma}(L^+E). \qquad (8)$$

Corollary 2. Suppose that $\gamma \leq \alpha\beta$ or that $\gamma > \alpha\beta$ and $\|L\| \leq \frac{\beta^2}{2(\gamma-\alpha\beta)}$. Then we have

$$\max_{\substack{E:\, E\in\mathbb{R}^{n\times k}\\ 1\leq k\leq n,\, \overline{\sigma}(E)\leq 1}} \overline{\sigma}(L^+E) \leq \frac{1}{\gamma}\|L^+\| = \frac{1}{\gamma\lambda_2},$$

where λ_2 denotes the second smallest eigenvalue of L.

Corollary 3. Suppose that $\gamma \leq \alpha\beta$ or that $\gamma > \alpha\beta$ and $\|L\| \leq \frac{\beta^2}{2(\gamma-\alpha\beta)}$. Then among all disturbance matrices $E \in \mathbb{R}^{n\times k}$ with $\overline{\sigma}(E) \leq 1$, the maximum from Corollary 2 is attained at $\mathbb{R}^{n\times 1} \ni E = v_2$, where v_2 is the (normalized) eigenvector of L corresponding to λ_2, that is, the Fiedler vector.

Proof: From Corollaries 1 and 2 it follows that it is sufficient to prove that the choice $E = v_2$ implies $\|F\|_\infty = 1/(\gamma\lambda_2)$, which readily follows from $L^+E = 1/\lambda_2 E$. □

Remark 1. In our previous paper Nakić et al. (2021), the main result covers only the case $E = e_i$, where e_i is the i^{th} canonical vector. In addition, owing to the weights η_i's and presence of $k \in \{1,\ldots,n\}$ exogenous signals introduced in Section 3, the present paper considers a more general problem setting than Nakić et al. (2021). For instance, this problem setting allows one to disturb several agents with the same signal (e.g., human-inflicted wind gusts generated by a single fan) while the impact of this signal on each agent might be different owing to different η_i's (e.g., the generated wind gusts disturb farther agents less whereas some agents might be impacted in the opposite direction).

Remark 2. For any E satisfying the assumptions from Section 3, Theorem 1 implies that maximal excitations of the system are obtained for constant disturbances because the H_∞ norms are attained in zero. However, among all E's from Corollary 3, it is enough to excite each agent with a constant disturbance and weights η_i's given by the Fiedler vector (i.e., $E = v_2$) for the maximal excursion w.r.t. equilibrium manifold. For example, if χ_i denotes the agent's position, the maximal departure from the consensus manifold is obtained by disturbing each agent with constant signals whose magnitudes and directions/signs are given by the respective components of the Fiedler vector.

5. NUMERICAL EXAMPLES

In this section, we consider three different cases in order to illustrate our new approach and show efficiency of the formula obtained in Theorem 1. In the examples, we vary the number of agents and the network parameters while the agent parameters are selected according to the quadrotor identification from Tolić et al. (2015b). In particular, using the notation from Section 3, we select $K_s = 5.2$, $T_s = 0.38$, $K = 0.6$ while parameters c_1 and c_1 are specified for each example below.

Owing to distinct, some of which are rather large, values of n and to avoid repetitiveness, the graph Laplacian matrices L are given by Algorithm 1 in both examples. Similarly, the edge weights are set to one in all graphs. In Algorithm 1, we use the `MATLAB` syntax when working with vectors.

Example 1. Let us illustrate an application of Theorem 1. Consider 20 agents and the graph illustrated in Figure 1. Regarding the disturbance matrix E, two cases are considered. The first case is with $E := E_1 = [e_2\ e_4\ e_6\ e_8]$ and the second case with $E := E_2 = [e_{11}\ e_{12}\ \ldots\ e_{16}]$, where e_i is the i^{th} canonical vector of dimension 20.

Let us also consider two cases of parameters, that is,

a) $(c_1, c_2) = (0.01, 0.3)$,

Algorithm 1 Construction of matrix L

Require: number of agents n
Ensure: the Laplacian matrix L
  $r = [1 : n/10, n/2 : 3n/5, 9n/10 : n] \in \mathbb{R}^{i_r}$
  $h = [n/4 : 2n/5, 7n/10 : 3n/4] \in \mathbb{R}^{i_h}$
  for $i = 1 : n-2$ **do**
    $L(i, i+1) = -1;$
    $L(i, n) = -1;$
  end for
  $L(1, n-1) = -1;$
  $L(n-1, n) = -1;$
  for $i = 1 : i_r$ **do**
    if $r(i) \sim= (n-1)$ **then**
      $L(r(i), r(i)+1) = 0;$
    else
      $L(1, n-1) = 0;$
    end if
  end for
  $L(h, n) = 0;$
  $L = L + L^T;$
  for $i = 1 : n$ **do**
    $L(i,i) = -\sum_{j=1}^{n}(L(i,j));$
  end for

b) $(c_1, c_2) = (1.5, 0.003)$.

For the case a), it holds that $\gamma - \alpha\beta = -0.3245 < 0$ meaning that the assumptions of Theorem 1 hold and that the maximum of transfer functions is attained at zero as shown in Figure 2 for two disturbance matrices E_1 and E_2.

For the case b), it holds that $\gamma - \alpha\beta = 4.6764 > 0$ and $\|L\| > \frac{\beta^2}{2(\gamma-\alpha\beta)}$. In other words, the assumptions of Theorem 1 do not hold and from this theorem we cannot claim that H_∞ norm is attained at zero. Furthermore, from Figure 2, we see that the maximum of our transfer function indeed is not attained at zero.

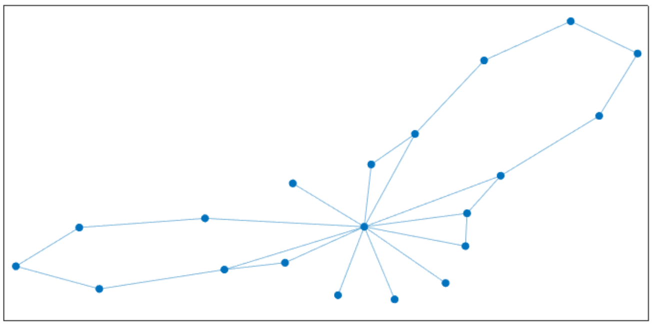

Fig. 1. Network topology in Example 1 where $n = 20$.

Example 2. Let us illustrate the computational benefits of our approach by considering large numbers of agents n. The group parameters are the same as in the case a) of Example 1 so that we can apply our new formula for H_∞ norm calculation with various number of agents.

The values $n = 500$, $n = 10000$ and $n = 40000$ are considered. The present approach uses the formula from Corollary 1 whilst the standard approach in `MATLAB` utilizes the function `hinfnorm` (with tolerance 0.001). Apparently, for the standard approach even dimension $n = 500$ is demanding while $n = 10000$ is not feasible. Table 1 shows the runtimes for $\|F\|_\infty$ computations with different matrices E_i, $i = 1, 2, \ldots, n-10$. In particular, the disturbance matrix E is defined by index i, where the i^{th} disturbance matrix is equal to $E_i = [e_i \; e_{i+i} \; \ldots \; e_{i+10}]$ with e_i being the i^{th} canonical vector of dimension n and $i \in \{1, 2, \ldots, n-10\}$. This means that in the case of 40000 agents, 39990 different $\|F\|_\infty$ norms are calculated, in the case of 10000 we have 9990 norm calculations, while in the case of 500 agents 490 computations are performed. Table 1 presents the average times for one $\|F\|_\infty$ calculation as well as the total computational times needed for all indices $i \in \{1, 2, \ldots, n-10\}$. The times given for one i represent the average times, but in the case $n = 10000$ it is not possible to evaluate `hinfnorm` for all indices i; hence, in this case (i.e., see the fourth column of Table 1) we include the time needed for $i = 1$.

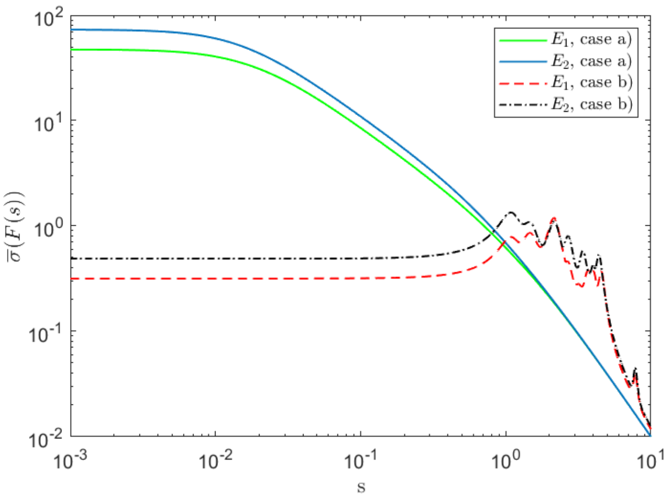

Fig. 2. Transfer functions for cases a) and b) in Example 1.

n	using (8) for one i	using (8) total time	`hinfnorm` for one i	`hinfnorm` total time
500	$1.7 \cdot 10^{-4}$ s	0.0822 s	28.3 s	230.9 min
10000	0.0232 s	3.87 min	34.4 h	n/a
40000	0.3633 s	242.1 min	n/a	n/a

Table 1. Runtime comparison.

As expected, the new approach that employs formula (8) leads to significant improvements in computational times. As seen from Table 1, our formula with $n = 40000$ and the standard approach with $n = 500$ have comparable execution times. These computations were carried out in `MATLAB` on a 64-bit Linux operating system and a machine with an AMD® Ryzen Threadripper™ processor with 64 CPUs, 128 threads and 256 GB DDR4 RAM.

6. CONCLUSION

This paper presents sufficient conditions that greatly improve agents-to-system H_∞-norm computations in MASs. Namely, these sufficient conditions allow one to focus merely on zero-frequency exogeneous signals when computing H_∞-norms of interest. In addition, the problem formulation and modeling are intentionally made rather general so as to encompass a wide range of MAS scenarios and disturbances. Undirected weighted topologies and second-order linear agent dynamics are considered. Numerical examples show that this approach is applicable to very large MASs (e.g., 40 000 agents).

The future research avenues include H_2-norm analyses, directed and time-varying topologies as well as more general MASs including agents with higher order dynamics.

ACKNOWLEDGEMENTS

The authors thank Prof. Andrej Jokić for fruitful discussions during early stages of this work.

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